Fundamental soliton derivation

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The following derivation is based on the one found in Chapter 5.2, titled "Fiber Solitons", of Govind P. Agrawal's book,"Nonlinear Fiber Optics".

The NLSE is:

$$i\frac{\partial A}{\partial z} = -\frac{i\alpha}{2}A + \frac{\beta_2}{2}\frac{\partial^2 A}{\partial T^2} - \gamma|A|^2 A, \qquad (3.1.1)$$

For a fiber, where attenuation is ignored, it's possible to normalize the NLSE in the following way:

$$U = \frac{A}{\sqrt{P_0}}, \quad \xi = \frac{z}{L_D}, \quad \tau = \frac{T}{T_0}, \qquad (5.2.1)$$

$$i\frac{\partial U}{\partial \xi} = \mathrm{sgn}(\beta_2)\frac{1}{2}\frac{\partial^2 U}{\partial \tau^2} - N^2|U|^2 U, \qquad (5.2.2)$$

Basically, amplitude is normalized to peak amplitude, distance is normalized to the dispersion length, $L_D = \frac{T_0^2}{|\beta_2|}$, and time is normalized to the pulse duration.

In the equation above, N^2 is given by:

$$N^2 = \frac{L_D}{L_{NL}} = \frac{\gamma P_0 T_0^2}{|\beta_2|}. \qquad (5.2.3)$$

Here $L_{NL} = \frac{1}{\gamma P_0}$ is the nonlinear length.

We can further normalize to the value of N,

$$u = NU = \sqrt{\gamma L_D}A. \qquad (5.2.4)$$

and assume that we have anormalous dispersion, so $sign(\beta_2) = -1$.

This produces the following version of the NLSE:

$$i\frac{\partial u}{\partial \xi} + \frac{1}{2}\frac{\partial^2 u}{\partial \tau^2} + |u|^2 u = 0, \qquad (5.2.5)$$

We wish to investigate if there exists a closed form expression for a field, u, which satisfies this equation.

One possible solution can be found using the following assumption:

Assume a separable solution of the form: $u = V(\tau)e^{i\phi(\xi)}$

Plug this into the NLSE (attenuation ignored):

$$\partial_\xi u = i\,\frac{1}{2}\,\partial_\tau^2 u \;+ i|u|^2 u$$

$$Vi\,(\partial_\xi \phi)\,e^{i\phi} = i\,\frac{1}{2}\,\partial_\tau^2\big(V(\tau)e^{i\phi(\xi)}\big) \;+ iV^3\,e^{i\phi}$$

$$Vi\,(\partial_\xi \phi)\,e^{i\phi} = i\,\frac{1}{2}\,e^{i\phi}(\partial_\tau^2 V) \;+ iV^3\,e^{i\phi}$$

Eliminate phase term and i:

$$V\,(\partial_\xi \phi) \;=\; \frac{1}{2}\,(\partial_\tau^2 V) \;+ V^3$$

Rearrange:

$$2V\,\left(\partial_\xi \phi\right) - 2V^3 \;=\; (\partial_\tau^2 V)$$

$$\partial_\tau^2 V = 2V(\,C - V^2\,)$$

$$\frac{\partial^2 V}{\partial \tau^2} = 2V(\,C - V^2\,)$$

Instead of a 2nd order differential equation, it would be nice to have a 1st order equation. To get this, multiply on both sides with 2 times the derivative of V:

$$2\frac{\partial V}{\partial \tau}\frac{\partial^2 V}{\partial \tau^2} = 2\frac{\partial V}{\partial \tau}2V(\,C - V^2\,)$$

Now exploit the product rule, which states that:

$$\left(fg\right)' = f'g + fg'$$

Choose:

$$f = g = \frac{\partial V}{\partial \tau}$$

$$fg' = \frac{\partial V}{\partial \tau}\frac{\partial^2 V}{\partial \tau^2}$$

$$f'g = \frac{\partial^2 V}{\partial \tau^2}\frac{\partial V}{\partial \tau}$$

This choice along with the product rule implies that

$$\left(\left(\frac{\partial V}{\partial \tau}\right)^2\right)' = 2\frac{\partial^2 V}{\partial \tau^2}\frac{\partial V}{\partial \tau},$$

allowing us to rewrite the differential equation as

$$2\frac{\partial V}{\partial \tau}\frac{\partial^2 V}{\partial \tau^2} = 2\frac{\partial V}{\partial \tau}2V(\,C - V^2\,)$$

$$\left(\left(\frac{\partial V}{\partial \tau}\right)^2\right)' = 4\frac{\partial V}{\partial \tau}\;V(\,C - V^2\,)$$

Multiply out the parenthesis:

$$\left(\left(\frac{\partial V}{\partial \tau}\right)^2\right)' = 4\frac{\partial V}{\partial \tau}\;V\,C - 4\frac{\partial V}{\partial \tau}\;V^3$$

Integrate on both sides:

$$\int\left(\left(\frac{\partial V}{\partial \tau}\right)^2\right)' d\tau = 4C\int\frac{\partial V}{\partial \tau}\;V\,d\tau - 4\int\frac{\partial V}{\partial \tau}\;V^3 d\tau$$

Consider

$$\int\frac{\partial V}{\partial \tau}\;V\,d\tau\,,$$

which can be integrated by parts by choosing

$$f' = \frac{dV}{d\tau},\qquad g = V$$

in:

$$\int f'g dx = [fg] - \int fg' dx$$

$$\int\frac{\partial V}{\partial \tau}\;V\,d\tau = [VV] - \int V\frac{\partial V}{\partial \tau}\;dx$$

Move the integral to the LHS from the RHS:

$$2\int\frac{\partial V}{\partial \tau}\;V\,d\tau = [VV]$$

$$\int\frac{\partial V}{\partial \tau}\;V\,d\tau = \frac{1}{2}[VV]$$

This simplifies the integration by parts:

$$\left(\frac{\partial V}{\partial \tau}\right)^2 = 4C\frac{1}{2}[V^2] - 4\int\frac{\partial V}{\partial \tau}\;V^3 d\tau$$

Now repeat for the other integral:

$$\int\frac{\partial V}{\partial \tau}\;V^3 d\tau$$

Choose $f' = \frac{\partial V}{\partial \tau}, \qquad g = V^3$

$$\int \frac{\partial V}{\partial \tau} V^3 dx = [VV^3] - \int V3V^2 \frac{\partial V}{\partial \tau}\ dx$$

$$\int \frac{\partial V}{\partial \tau} V^3 dx = [V^4] - 3\int V^3 \frac{\partial V}{\partial \tau}\ dx$$

Move integral to LHS from RHS:

$$4\int \frac{\partial V}{\partial \tau} V^3 dx = [V^4]$$

$$\int \frac{\partial V}{\partial \tau} V^3 dx = \frac{1}{4}[V^4]$$

Now the 2nd order equation has been replaces with a 1st order equation:

$$\left(\frac{\partial V}{\partial \tau}\right)^2 = 4C\frac{1}{2}[V^2] - 4\ \frac{1}{4}[V^4]\ + K$$

Here, K is an unknown integration constant, whose value depends on the boundary conditions. These must be chosen to obtain a physically meaningful pulse (finite energy, decreasing amplitude at infinity etc.)

$$\left(\frac{\partial V}{\partial \tau}\right)^2 = 2CV^2 - V^4\ + K$$

$$\left(\frac{\partial V}{\partial \tau}\right)^2 = V^2(2C - V^2) + K$$

Demand that both the derivatives and the value vanishes at infinity:

$$\left.\frac{\partial V}{\partial \tau}\right|_{\tau=0} = 0$$

$$\left.\frac{\partial V}{\partial \tau}\right|_{\tau=\pm\infty} = 0$$

$$V(\pm\infty) = 0$$

This requirement ensures that

$$0\ = 0 * (2C - 0) + K$$
$$K = 0$$

Implying that

$$\left(\frac{\partial V}{\partial \tau}\right)^2 = V^2(2C - V^2)$$

Demand that V peaks with a value of 1 at tau=0:

$$\left.\frac{\partial V}{\partial \tau}\right|_{\tau=0} = 0$$

$V(0) = 1$

This ensures that

$0 = (2C - 1)$

Leading to:

$$C = \partial_\xi \phi = \frac{1}{2}$$

And

$u = V(\tau)e^{i\xi/2}$

The differential equation now reduces to:

$$\left(\frac{\partial V}{\partial \tau}\right)^2 = V^2(1 - V^2)$$

Can we find some elementary function, which satisfies this differential equation?

Consider:

$$\frac{d}{dx}\operatorname{sech} x = -\tanh x \operatorname{sech} x$$

Compute the square on both sides:

$$\left(\frac{d}{dx}\operatorname{sech}(x)\right)^2 = \tanh^2(x)\operatorname{sech}^2(x)$$

Now exploit that

$$\operatorname{sech}^2 x = 1 - \tanh^2 x$$

To rewrite:

$$\left(\frac{d}{dx}\operatorname{sech}(x)\right)^2 = \operatorname{sech}^2(x)\,(1 - \operatorname{sech}^2 x)$$

Compare with our differential equation:

$$\left(\frac{\partial V}{\partial \tau}\right)^2 = V^2(1 - V^2)$$

We see that

$V = \operatorname{sech}(\tau)$

Solves the equation. Therefore, the soliton that follows the separable form,

$u = V(\tau)e^{i\phi(\xi)}$,

is described by:

$u(\tau, \xi) = \text{sech}(\tau)\ e^{i\xi/2}$

Note that this solution only describes a soliton whose amplitude is chosen so anormalous dispersion and self-phase modulation cancel each other exactly, causing the pulse envelope to be constant for all distances (N=1). For a more intense pulse (N=2,3,4,...) the simple ansatz, $u = V(\tau)e^{i\phi(\xi)}$, does not work and we need a more intricate expression, which causes the evolution of the soliton will be more complicated.

For example, for N=2, it can be shown that the following pulse will solve the normalized NLSE with anormalous dispersion and no attenuation:

$$u(\xi, \tau) = \frac{4[\cosh(3\tau) + 3\exp(4i\xi)\cosh(\tau)]\exp(i\xi/2)}{[\cosh(4\tau) + 4\cosh(2\tau) + 3\cos(4\xi)]}. \qquad (5.2.23)$$